MA 3046 - Matrix Analysis
Introduction to Decomposition and Projections

As we have already mentioned, most students receive their first real introduction to vectors in some physics class. Moreover, usually among the vectors first discussed there are forces and velocities, and one of the first applications of these vectors is to consider the work done moving a "particle" along a path where the force involved ($\mathbf{f}$) does not exactly align with the direction of travel ($\mathbf{v}$). (Figure 1)

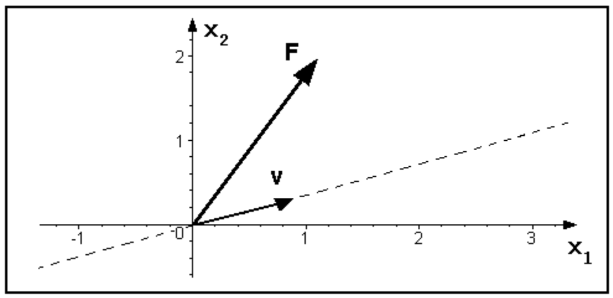

Figure 1

In such instances, students learn that work only done by the *component* of the force that lies in direction of travel, i.e. the "hollow" vector shown in Figure 2.

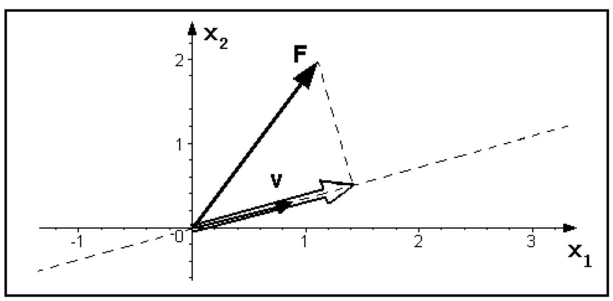

Figure 2

But, in this instance, basic trigonometry (Figure 3)

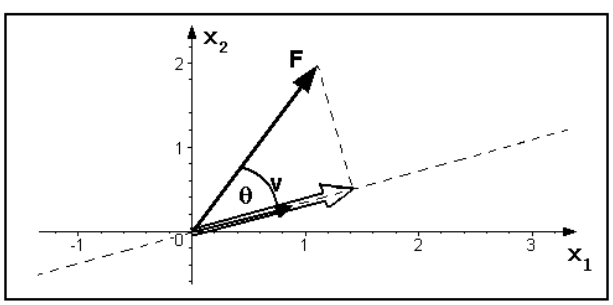

Figure 3

implies that this component has magnitude equal to $\| \mathbf{F} \| \cos(\theta)$, where $\| \cdot \|$ denotes the usual (Euclidean) length of any vector. Equivalently, using the dot product, the length (magnitude) of this component is

$$\| \mathbf{F} \| \cos(\theta) = \frac{\mathbf{F} \cdot \mathbf{v}}{\| \mathbf{v} \|}$$

Moreover, since any vector is completely defined by its magnitude and direction, we can express this component vector by simply multiplying the above magnitude by a unit vector in the direction of the component, which, of course, is also the direction of $\mathbf{v}$. This last observation results in the expression for the component of

$$\frac{\mathbf{F} \cdot \mathbf{v}}{\| \mathbf{v} \|} \, \frac{\mathbf{v}}{\| \mathbf{v} \|} = \frac{\mathbf{F} \cdot \mathbf{v}}{\| \mathbf{v} \|^2} \; \mathbf{v}$$

But, if we express $\mathbf{F}$ and $\mathbf{v}$ in terms of column vectors, then the component of $\mathbf{F}$ in the direction of $\mathbf{v}$ can also be written

$$\frac{\mathbf{F}^T \, \mathbf{v}}{\mathbf{v}^T \, \mathbf{v}} \;\; \mathbf{v} \;=\; \frac{\mathbf{v}^T \, \mathbf{F}}{\mathbf{v}^T \, \mathbf{v}} \;\; \mathbf{v} \;=\; \mathbf{v} \;\; \frac{\mathbf{v}^T \, \mathbf{F}}{\mathbf{v}^T \, \mathbf{v}} \;=\; \frac{\mathbf{v} \, \mathbf{v}^T}{\mathbf{v}^T \, \mathbf{v}} \;\; \mathbf{F}$$

where we are know the dot product is commutative, and we are allowed to commute the order of the other terms since $\mathbf{v}^T \, \mathbf{F}/\mathbf{v}^T \, \mathbf{v}$ is a scalar, and scalars and vectors commute.

But now, if we step back and view this from a more linear algebra perspective, all we have really done is decomposed $\mathbf{F}$ into sum of two components - one parallel to the line defined by $\mathbf{v}$, and one perpendicular to that line. (Figure 4)

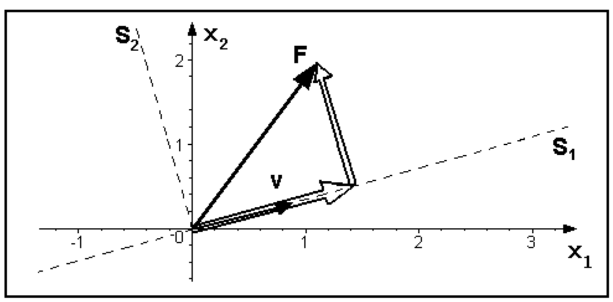

Figure 4

Moreover, that line defined by $\mathbf{v}$ represents no more and no less than precisely the subspace ($\mathbf{S_1}$) *spanned* by $\mathbf{v}$. Hence what we are doing is very little different from the problem of finding coordinates in terms of another (orthogonal) basis.

Lastly, we can also interpret the component of $\mathbf{F}$ in the direction of $\mathbf{v}$ as precisely the "shadow" that would be cast by $\mathbf{F}$ if we were to shine a light down on (i.e. perpendicular to) the line (subspace) $\mathbf{S_1}$. (Figure 5) (or equivalently, if we shine the light parallel to (along the direction of) the line (subspace) $\mathbf{S_2}$. In view of this interpretation, and because this is virtually identical to what we do when we project a film onto a screen, we may now also call the component of $\mathbf{F}$ in the direction of $\mathbf{v}$ the *projection* of $\mathbf{F}$ onto the subspace spanned by $\mathbf{v}$, or the projection of $\mathbf{F}$ onto $\mathbf{S_1}$.

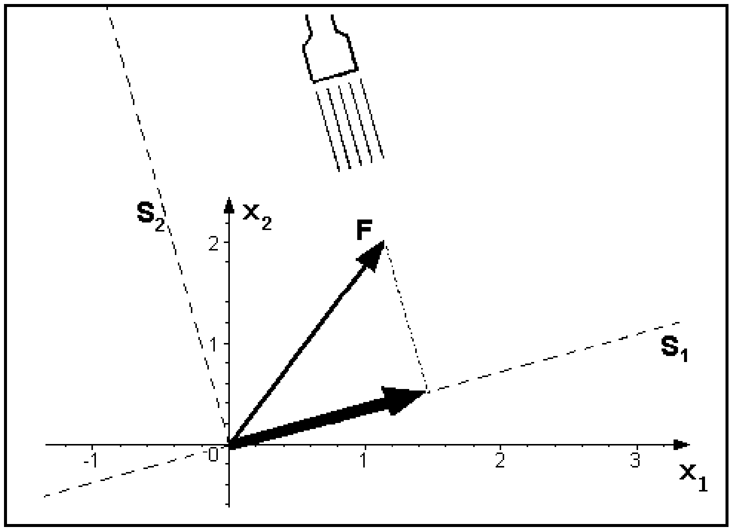

Figure 5

In fact we can, and will, even extend this notion to the case where the subspaces (and hence their bases) are not orthogonal. For example, in Figure 6, the vector $\mathbf{w}$ remains, geometrically, precisely the "shadow" that would be cast by $\mathbf{F}$ if we now shine our light parallel to $\tilde{\mathbf{S}}_2$. Therefore, in this case, again, we will call $\mathbf{w}$ the projection of $\mathbf{F}$ onto $\tilde{\mathbf{S}}_1$ along $\tilde{\mathbf{S}}_2$.

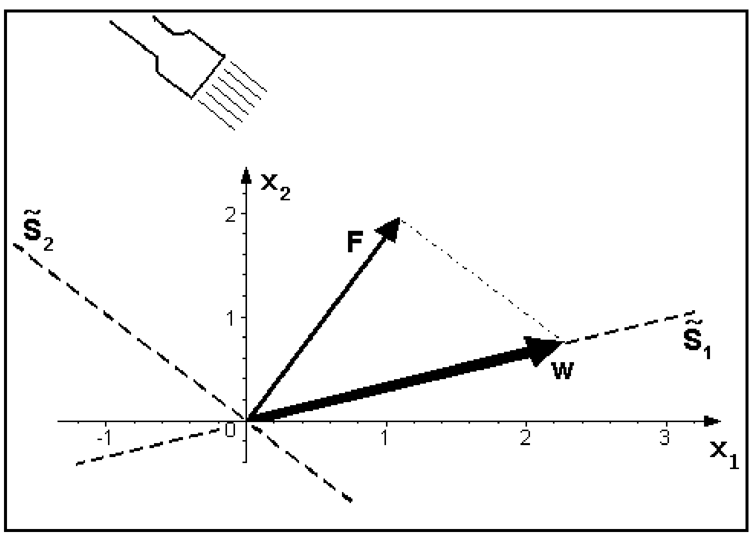

Figure 6

But finally, we also observe that $\mathbf{w}$ represents precisely the vector in $\tilde{\mathbf{S}}_1$ that will result if we decompose the vector $\mathbf{F}$ into "components" along the directions of $\tilde{\mathbf{S}}_1$ and $\tilde{\mathbf{S}}_2$ as shown (Figure 7), i.e.

$$\mathbf{F} = \mathbf{u} + \mathbf{w}$$

where $\mathbf{w}$ lies in (the direction of) $\tilde{\mathbf{S}}_1$ and $\mathbf{u}$ lies in (the direction) of $\tilde{\mathbf{S}}_2$. So again, we see a relationship between these geometrical projections and the question of components.

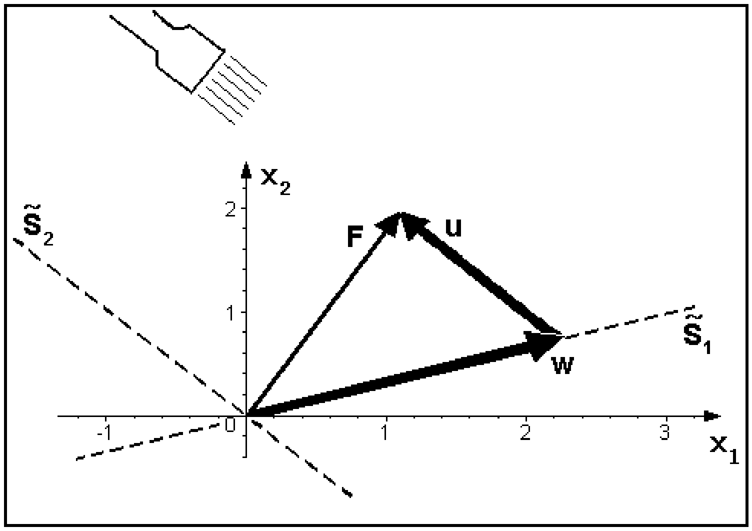

Figure 7